FINAL PROJECT
OPEN PROBLEMS IN NUMBER THEORY
SPRING 2018, TEL AVIV UNIVERSITY

Instructions: You may not cooperate with anyone on the project

$\boxed{\text{Due Date: July 18, 2018.}}$

Put the completed assignment in my mailbox, or mail to rudnick@post.tau.ac.il

Exercise 1. *Let*

$$N_3(R) = \#\mathbb{Z}^3 \cap B(0,R)$$

be the number of lattice points in the 3-dimensional ball of radius R. Show that

$$N_3(R) = \frac{4\pi}{3}R^3 + O(R^{3/2}).$$

Exercise 2. *Let $N \gg 1$. Show that for any triple $\{d_1, d_2, d_3\}$ of divisors of N, so that $\sqrt{N} \le d_1 < d_2 < d_3 \le N$, we must have $d_3 - d_1 \ge N^{1/6}$. That is, the three integer points $(d_j, N/d_j)$ on the hyperbola $xy = N$ cannot lie in an arc of diameter $\ll N^{1/6}$.*

Hint: One way to do this is to prove that

$$\prod_{1 \le i < j \le 3} (d_j - d_i) \ge N^{1/2}.$$

Exercise 3. *Here is an example showing why one needs the exponent to be bigger than 1 in the abc conjecture: Let $A_k = 3^{2^k} - 1$, $B_k = 1$, $C_k = 3^{2^k}$. Show that for all $k \ge 2$,*

$$C_k \ge \frac{2^{k+1}}{3}\operatorname{rad}(A_k B_k C_k)$$

and hence that

$$C_k \gg R_k \log R_k,$$

where $R_k := \operatorname{rad}(A_k B_k C_k)$.

Hint: Show that 2^{k+2} divides $3^{2^k} - 1$.

Exercise 4. *Show that the abc conjecture implies Pillai's conjecture: each positive integer occurs at most finitely many times as a difference of perfect powers, i.e. for each $k \ge 1$, there are at most finitely many solutions of $x^m - y^n = k$ with $x, y, m, n \ge 2$.*

Exercise 5. *Prove the polynomial Catalan conjecture: There are no consecutive perfect powers of positive degree in $\mathbb{C}[x]$, i.e. no solutions of $f(x)^m - g(x)^n = 1$ with $f, g \in \mathbb{C}[x]$, $m, n \ge 2$ and $\deg f > 0$.*